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ON THE SOLVABILITY OF NON-LINEAR SHALLOW SHELL EQUILIBRIUM PROBLEMS*

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The proof is obtained for a theorem of the existence of the solution of a shallow shell equilibrium problem under the most general edge fixing conditions. For instance, it is sufficient that the fixing conditions should ensure that there are no displacements of the shell as a rigid whole. However, certain constraints on the magnitude of the tangential external forces must be introduced here.

A strict mathematical proof of the solvability of equilibrium problems in non-linear shallow shell theory and the convergence of different approximate methods for solving it is known under quite broad assumptions relative to the shell geometry, the magnitude of the load, and the boundary conditions /1, 2/. The main statement of the proof of the corresponding theorem is either to obtain an a priori estimate of the solution or to obtain an estimate of the shell total energy functional. Gaps that exist at this time in the mathematical theory of boundary value problems for shallow shell equations in displacements are due to the fact that the method developed in /1, 2/ for obtaining the estimate requires that the tangential displacements of the shell middle surface u, v be given on a "substantial" part of the boundary (in particular, on the whole boundary for a convex shell).

1. In order not to complicate the main idea of the proof with details, we will consider the simplest and most widespread modification of the non-linear theory of isotropic, homogeneous, shallow shells of constant thickness $2h$ in displacements, in which the geometry of the shell middle surface is identified with a plane /3/. The equilibrium equations have the form

$$\begin{aligned}
&D\nabla^4 w + N_1(k_1 - w_{xx}) + N_2(k_2 - w_{yy}) - 2N_{12}w_{xy} - F_3 = 0 \qquad (1.1)\\
&\nabla^2 u + \frac{1+\mu}{1-\mu}(u_y + v_x)_x + \frac{2}{1-\mu}((k_1 w)_x + w_x w_{xx} +\\
&\quad \mu(k_2 w)_x + \mu w_y w_{xy}) + w_y w_{xy} + w_x w_{yy} + F_1 = 0\\
&\nabla^2 v + \frac{1+\mu}{1-\mu}(u_y + v_x)_y + \frac{2}{1-\mu}((k_2 w)_y + w_y w_{yy} +\\
&\quad \mu(k_1 w)_y + \mu w_x w_{xy}) + w_x w_{xy} + w_y w_{xx} + F_2 = 0\\
&N_1 = Eh(1-\mu^2)^{-1}(\varepsilon_1 + \mu\varepsilon_2),\quad N_2 = Eh(1-\mu^2)^{-1}(\varepsilon_2 +\\
&\quad \mu\varepsilon_1)\\
&N_{12} = {}^1/_2 Eh(1+\mu)^{-1}\varepsilon_{12},\quad \varepsilon_1 = u_x + k_1 w + {}^1/_2 w_x^2\\
&\varepsilon_2 = v_y + k_2 w + {}^1/_2 w_y^2,\quad \varepsilon_{12} = u_y + v_x + w_x w_y
\end{aligned}$$

Here w is the normal displacement of the shell middle surface, u, v are the principal curvatures, E, μ are the elastic constants, and F_i are the external loads.

Let the shell planform occupy a domain Q with piecewise-smooth boundaries ∂Q such that the Sobolev embedding theorems /4/ are satisfied for functions defined in Q.

We will indicate the minimum necessary conditions for shell supports for which the main theorem will be obtained.

Let $w(x_i, y_i) = 0$ at three points $(x_i, y_i),\ i = 1, 2, 3$ of the domain Q that do not lie on one line. Moreover, on the part of the boundary ∂Q_1 which can in fact be missing, $w|_{\partial Q_1} = 0$. The subspace of functions from $C^{(4)}(Q)$ satisfying these conditions will be denoted by $C_1^{(4)}$.

For the tangential displacements u, v such boundary conditions will be the minimum necessary so that the Korn inequality for the plane problem of elasticity theory /5, 6/ is satisfied for them

$$\int (u^2 + v^2 + u_x^2 + u_y^2 + v_x^2 + v_y^2)\,dx\,dy \leqslant m\int (u_x^2 + v_y^2 + (u_y + v_x)^2)\,dx\,dy$$

(here and henceforth, the domain of integration Q is not indicated).

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The sufficient conditions for this are stated in /5, 6/. In particular, the Korn inequality will hold if on some part of the boundary of non-zero length ∂Q_2

$$u, v \mid_{\partial Q_2} = 0 \tag{1.2}$$

To be specific, we shall consider this condition to be satisfied.

The set of vector functions $\mathbf{u}^* (u, v)$, each of whose components lies in $C^{(2)}(Q)$ satisfying condition (1.2) will be denoted by $C_1^{(2)}$.

We shall consider the remaining boundary conditions, not indicated here, as natural, i.e., obtained directly from the variational formulation of the problem. Since they are well-known, we shall not write them down. Some additional conditions on the boundary are certainly possible, but they do not affect the course of the proof.

2. Let us introduce energetic spaces. Let H_1 be the subspace $W_2^{(1)}(Q) \times W_2^{(1)}(Q)$ obtained by closing the set $C_1^{(2)}$ therein. The Korn inequality ensures that there is an equivalent norm in H_1

$$\| \mathbf{u} \|_{H_1}^2 = \frac{Eh}{2(1-\mu^2)} \int (e_1^2 + e_2^2 + 2\mu e_1 e_2 + {}^1/_2 (1-\mu) e_{12}^2) \, dx \, dy$$

$$(e_1 = u_x, \; e_2 = v_y, \; e_{12} = u_y + v_x)$$

The space H_2 is the subspace $W_2^{(2)}(Q)$ obtained by closing the set of functions $C_1^{(4)}$ therein. The equivalent norm /7/

$$\| w \|_{H_2}^2 = {}^1/_2 D \int ((\nabla^2 w)^2 + 2(1-\mu)(w_{xx} w_{yy} - w_{xy}^2)) \, dx \, dy$$

is defined in the space.

Norms in the spaces H_i induce scalar products which we denote by $(a, b)_{H_i}$. We denote the space $H_1 \times H_2$ by H and the variation of the function f by δf.

We will call the vector function $\mathbf{u} \in H$ satisfying the integrodifferential equation

$$\int (M_1 \delta \varkappa_1 + M_2 \delta \varkappa_2 + 2M_{12} \delta \chi + N_1 \delta \varepsilon_1 + N_2 \delta \varepsilon_2 + 2N_{12} \delta \varepsilon_{12}) \, dx \, dy = \tag{2.1}$$

$$\int (F_1 \delta u + F_2 \delta v + F_3 \delta w) \, dx \, dy + \int_{\partial Q} (f_1 \delta u + f_2 \delta v + f_3 \delta w) \, ds$$

$$M_1 = D(\varkappa_1 + \mu \varkappa_2), \quad M_2 = D(\varkappa_2 + \mu \varkappa_1), \quad M_{12} = D(1 - \mu) \chi$$

$$\varkappa_1 = -w_{xx}, \quad \varkappa_2 = -w_{yy}, \quad \chi = -w_{xy}$$

the generalized solution of the shallow shell equilibrium problem, where the vector function $\delta \mathbf{u} = (\delta u, \delta v, \delta w)$ is arbitrary, and f_i are the external loads applied to the shell endfaces.

We note that assignment of an appropriate load f_i is not required on the part of the boundary where any of the components δu, δv, δw equals zero. But we shall not extract this part in writing the line integral since the appropriate part of the integral equals zero if f_i is predetermined zero, say.

It is sufficient for correctness in determining the generalized solution that the right-hand side of (2.1) be a continuous functional in $\delta \mathbf{u}$ in H. In turn, it is sufficient for this that

$$f_i \in L_p(\partial Q), \quad F_i \in L_p(Q), \quad i = 1, 2, \quad p > 1, \quad \text{a } f_3, F_3$$

be finite sums of δ - functions and functions in ∂Q and Q, respectively, from L_1. We shall call such a class of loads H^*.

For a generalized solution to exist it is necessary that the load belong to the class H^*.

If it exists, the classical solution of the problem will indeed be a generalized solution in the above-mentioned sense. In the general case the converse is not true.

Underlying the proof of the theorem on solvability is the fact that stationary points of the shell total energy functional

$$I(\mathbf{u}) = \| w \|_{H_2}^2 + {}^1/_2 \int (N_1 \varepsilon_1 + N_2 \varepsilon_2 + 2N_{12} \varepsilon_{12}) \, dx \, dy -$$

$$\int (F_1 u + F_2 v + F_3 w) \, dx \, dy - \int_{\partial Q} (f_1 u + f_2 v + f_3 w) \, ds$$

are genealized solutions of the shell equilibrium problem.

3. The structure and fundamental properties of the functional I remain the same for the boundary conditions under consideration as for the boundary conditions of the problem from /1, 2/. Consequently, it is sufficient to prove that

$$I(\mathbf{u}) \to \infty, \quad \text{if} \quad \| \mathbf{u} \|_H \to \infty \tag{3.1}$$

to prove the existence theorem

Let us prove this estimate.

To shorten the writing we will introduce an element $g \in H$ by the relationship

$$(g, u)_H = \int (F_1 u + F_2 v + F_3 w)\, dxdy + \int_{\partial Q} (f_1 u + f_2 v + f_3 w) ds$$

by using the Riesz theorem on a continuous linear functional in Hilbert space and the properties of an external load (the load belongs to the class H^*).

Then the functional I takes the form

$$I(u) = \| w \|_{H_2}^2 + {}^1/_2 \int (N_1 \varepsilon_1 + N_2 \varepsilon_2 + 2N_{12} \varepsilon_{12})\, dx\, dy - (g, u)_H$$

As in /2/, we will further consider the functional I on an "ellipsoid" $T(R)$ of the space H obtained by deformation of a unit sphere S of the space H with centre at zero in the following way. An element (cR^2u, cR^2v, Rw), where $c > 0$ is a certain number that will be determined later, on the ellipsoid $T(R)$ corresponds to the element (u, v, w) lying on the sphere S. For large R it is included within a sphere of radius R^2 and contains a sphere of the space H of radius R with centre at zero within itself. Consequently, to prove (3.1) it is sufficient to show that

$$I(u) \to \infty, \quad u \in T(R) \quad R \to \infty$$

We note that from the mechanical viewpoint, the consideration of the energy functional I on $T(R)$ results in an increase in the relative fraction of the energy of the part that is formed because of tangential displacements.

Let us divide the sphere S of the space H into two parts S_1 and S_2. Let the inequality

$$\| u^* \|_{H_1} \geqslant {}^1/_2, \quad u^* = (u, v) \tag{3.2}$$

be satisfied on S_1.

We consider the positive form

$$\int (N_1 \varepsilon_1 + N_2 \varepsilon_2 + 2N_{12} \varepsilon_{12})\, dxdy \tag{3.3}$$

It is homogeneous in R for mapping on the ellipsoid $T(R)$ The degree of its homogeneity is 4 while the degree of homogeneity of the remaining terms is not higher than two.

On $S_1 \| w \|_{H_2}^2 \leqslant {}^1/_2$. By virtue of the Sobolev embedding theorem /4/ here

$$\int (w_x^4 + w_y^4)\, dxdy \leqslant m = \text{const}$$

Since the integrand in (3.3) is a positive-definite form in the components of ε, by virtue of (3.2) a constant $c > 0$ can always be selected on S_1 such that

$$\int (N_{1c} \varepsilon_{1c} + N_{2c} \varepsilon_{2c} + 2N_{12c} \varepsilon_{12c})\, dxdy \geqslant 1$$

where the subscript c denotes that cu, cv are substituted in place of u, v in the appropriate expressions. We will determine such a c. In this case the inequality

$$\int (N_1 \varepsilon_1 + N_2 \varepsilon_2 + 2N_{12} \varepsilon_{12})\, dxdy \geqslant R^4 \text{ on } T_1(R)$$

is satisfied in the image of S_1 in $T(R)$. And thereby on $T_1(R)$ for large R

$$I(u) \geqslant {}^1/_2 R^4 \tag{3.4}$$

We will examine $I(u)$ on $S_2 = S/S_1$. The following inequalities hold:

$$I(u) \geqslant \| w \|_{H_2}^2 - (g, u)_H \geqslant \| w \|_{H_2}^2 - (g^*, u^*)_{H_1} - \| g_3 \| \| w \|_{H_2}$$

$$(g^*, u^*)_{H_1} = \int (F_1 u + F_2 v)\, dx\, dy + \int_{\partial Q} (f_1 u + f_2 v)\, ds$$

Under the substitution $(u, v, w) \to (cu, cv, w)$

$$I_c(u) \geqslant \| w \|_{H_2}^2 - c \| g^* \|_{H_1} \| u^* \|_{H_1} - \| g_3 \|_{H_2} \| w \|_{H_2}$$

Consequently

$$I(u) \geqslant {}^1/_2 R^2 (1 - c \| g^* \|_{H_1}) - {}^1/_2 \| g_3 \|_{H_2} R$$

on $T_2(R)$ is the image of S_2 in $T(R)$.

If the external tangential loads are such that

$$c\,\|g^*\|_{H_1} \leqslant {}^1/_2 \tag{3.5}$$

then for sufficiently large R we have $I(\mathbf{u}) \geqslant {}^1/_5 R^2$ on $T_2(R)$.

This last inequality, together with estimate (3.3), completes the proof of (3.1).

As in /1/, the following fundamental theorem therefore results.

Theorem 1. Let the tangential loads on the shell be sufficiently small, i.e., the inequality (3.5) satisfied, and all the loads belong to the class H^*. In this case

a) there exists at least one generalized solution of the equilibrium problem of a shell with finite energy;

b) the sequence $\mathbf{u}_n$ minimizing the functional $I(\mathbf{u})$ contains a subsequence that converges strongly in H to the generalized solution of the problem;

c) the system of equations of the approximate solution of the problem by the Ritz method (and thereby by the Bubnov-Galerkin method also) is solvable at each stage and contains a subsequence which converges strongly in H to the generalized solution.

We will not present the proof of the theorem here since after the proof of the relation (3.1) the rest of the proof is carried over from /1/ word for word. The remaining results from /1/ are also carried over without any difficulties in this case.

Remark. Theorem 1 holds without any constraints for shells to which only the normal loads f_3, F_3 are applied.

4. We will examine the question of the magnitude of the external tangential loads in greater detail. Their estimate, which is sufficient for the problem to be solvable, can be obtained most conveniently within the framework of that formulation of the problem when the tangential displacements are expressed in terms of w by means of (1.1). Then the functional I depends only on w. It is shown in /1/ that even in this case the stationary points of the functional $I(w)$ yield a generalized solution of the problem under consideration in the formulation mentioned. It is shown there also that the solution $\mathbf{u}^* = (u, v)$ of the equation is separated into the sum $\mathbf{u}^* = \mathbf{u}_0^* + \mathbf{u}_1^* + \mathbf{u}_2^*$, where the subscript equals the degree of homogeneity in the variable w for the generalized solution of system (1.1). The part $\mathbf{u}_2^* = (u_2, v_2)$ which is determined by the equation

$$\int (N_1\varepsilon_1 + N_2\varepsilon_2 + 2N_{12}\varepsilon_{12})\,dxdy = 0$$

plays the most important role in obtaining the necessary estimate.

The estimate

$$\int u_{2x}^2\,dx\,dy \leqslant \int w_x^4\,dx\,dy, \quad \int v_{2y}^2\,dx\,dy \leqslant \int w_y^4\,dx\,dy \tag{4.1}$$
$$\int (u_{2y} + v_{2x})^2\,dx\,dy \leqslant \int w_x^2 w_y^2\,dx\,dy$$

can be obtained from this equation.

For a given modification of the solution of the problem of minimizing the energy functional, an estimate of part of the functional must be made on the sphere $\|w\|_{H_2} = 1$ /1/. In order for the functional $I(w)$ to be increasing, it is sufficient to show that the inequality

$$\|w\|_{H_2}^2 - \int (F_1u_2 + F_2v_2)\,dx\,dy - \int_{\partial Q} (f_1u_2 + f_2v_2)\,ds \geqslant \alpha = \text{const} > 0 \tag{4.2}$$

is satisfied on the sphere $\|w\|_{H_2} = 1$.

Taking account of the Korn inequality as well as (4.1), the integral terms can be estimated here as follows:

$$\begin{aligned}
A \equiv &\left|\int (F_1u_2 + F_2v_2)\,dx\,dy + \int_{\partial Q} (f_1u_2 + f_2v_2)\,ds\right| \leqslant \\
&m_1(\|F_1\|_{p,Q} + \|F_2\|_{p,Q} + \|f_1\|_{p,Q} + \\
&\|f_2\|_{p,Q})\left(\int (u_{2x}^2 + v_{2y}^2 + 2(u_{2y} + v_{2x})^2)\,dx\,dy\right)^{1/2} \equiv \\
&m_1B\left(\int (u_{2x}^2 + v_{2y}^2 + 2(u_{2y} + v_{2x})^2)\,dx\,dy\right)^{1/2} \leqslant \\
&m_1B\left(\int (w_x^2 + w_y^2)^2\,dx\,dy\right)^{1/2}, \quad p > 1
\end{aligned}$$

where $\|g\|_{p,Q}$ is the norm of g in $L_p(Q)$.

By virtue of the Sobolev embedding theorem in $W_2^{(2)}(Q)$

$$\left(\int (w_x^2 + w_y^2)^2\,dx\,dy\right)^{1/2} \leqslant m_2 \int \left((\nabla^2 w)^2 + 2(1-\mu)(w_{xx}w_{yy} - w_{xy}^2)\right)dx\,dy$$

Consequently

$$A \leqslant 2m_1m_2B \, \| w \|_{H_2}{}^2/D$$

Finally, we have from (4.2)

$$\| F_1 \|_{p,Q} + \| F_2 \|_{p,Q} + \| f_1 \|_{p,Q} + \| f_2 \|_{p,Q} \leqslant D(1-\alpha)/(2m_1m_2)$$

We note that for a similar change in the domain Q (together with the domain in which the boundary conditions are specified) with similarity coefficient l, the constants m_i depend on l as follows:

$$m_1 = m_1{}^\circ l, \quad m_2 = m_2{}^\circ l^2$$

All the results obtained above remain valid even for the modification of the non-linear shallow shell theory considered in /2/, including also for the boundary conditions that include shell elastic support conditions.

5. We will make just one more remark. Estimation of the total energy functional does not ensure on a priori estimate of all the generalized solutions of the problem. Additional sufficient conditions can be indicated that ensure such a constraint. This is the upper limit of the integral

$$\int (k_1{}^2 + k_2{}^2) dxdy$$

The constant, which should be smaller than this integral, depends on the ratio between the shell thickness and its characteristic dimension and the first natural frequency of the transverse vibrations of a linear shell.

In addition, we note that geometric connections can generally be removed from the tangential displacements u, v. Here free "stiff" displacements of the form

$$u_0 = a + dy, \; v_0 = b - dx \tag{5.1}$$

of the shell appear, where a, b, d are arbitrary constants. If these displacements are added to those already existing in the shell, its stresses and strains do not change here. It can be shown that selfequilibration of the tangential load is required for the problem to be solvable in this case. Namely, that for all the constants a, b, d

$$\int (F_1u_0 + F_2v_0) \, dx \, dy + \int_{\partial Q} (f_1u_0 + f_2v_0) \, ds = 0 \tag{5.2}$$

This condition means that the principal vector and moment (planar!) of the tangential forces equal zero.

To obtain the existence theorem in this case is is necessary to introduce the factor-space H_{11} in place of H_1, which has the same norm as in H_1, and a vector-function of the form (5.1) is its kernel.

Reasoning similar to that performed in /7/ results in the following theorem.

Theorem 2. Let a load acting on a shell belong to the class H^* and let condition (3.5) be satisfied. Then for a generalized solution of the shallow shell equilibrium problem to exist when there are no geometric connections in the tangential variables u, v it is necessary and sufficient that the tangential load be selfequilibrated, i.e, that condition (5.2) be satisfied.

We note that the solution of the equilibrium problem for a shell not at all free of geometric connections has no meaning in a similar formulation.

The reasons are as follows. Real free displacements of a shell as a rigid whole cause strains and stresses in the shell in this modification of the equations. On the other hand if we proceed formally and introduce stiff displacements as functions that convert the quadratic part of the energy to zero, then a set of "stiff" displacements appears on which the work of the external forces must equal zero. But these integral conditions on the forces have no mechanical meaning. Moreover, tangential stresses different from zero appear in the shell in such stiff displacements.

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FORCED VIBRATIONS OF A PIEZOCERAMIC CYLINDRICAL SHELL WITH LONGITUDINAL POLARIZATION*

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Forced vibrations of a circular cylindrical piezoceramic shell with longitudinal polarization caused by an electric load applied to electrodes on the shell edge are considered. A numerical computation is performed by the partition method for the electroelastic state, and values of the coefficient of electromechanical coupling obtained by different formulas are compared.

1. We select the system of orthogonal curvilinear dimensionless coordinates ξ, φ such that the ξ-line coincides with the generatrix and the φ-line with the directrix of the cylinder.

We write down the system of equations for the electroelastic shell under consideration in the selected coordinates and we omit here certain equations not used below.

The equilibrium equations

$$dT_{1n}/d\xi - nS_{12n} + \lambda u_n = 0 \tag{1.1}$$

$$T_{2n} + \varepsilon^2 dN_{1n}/d\xi - \varepsilon^2 nN_{2n} + \lambda w_n = 0$$

$$dS_{12n}/d\xi + nT_{2n} - N_{2n} + \lambda v_n = 0 \tag{1.2}$$

$$N_{1n} = dG_{1n}/d\xi \tag{1.3}$$

The electroelasticity relationships

$$T_{1n} = \varepsilon_{1n} + \nu_2\varepsilon_{2n} - E_{1n}, \quad T_{2n} = \sigma(\varepsilon_{2n} + \nu_1\varepsilon_{1n}) - c_{12}E_{1n} \tag{1.4}$$

$$S_{12n} = S_{21n} = (\omega_n - d_{15}n_{22}c_2^{-1}E_{2n})/s_{44}{}^E n_{22} \tag{1.5}$$

$$G_{1n} = -\varepsilon^2\varkappa_{1n} \tag{1.6}$$

$$D_{1n} = \varepsilon_{33}{}^T(c_2d_{31})^{-1}E_{1n} + T_{2n} + d_{33}(d_{31})^{-1}T_{1n} \tag{1.7}$$

$$D_{2n} = \varepsilon_{11}{}^T(c_2d_{31})^{-1}E_{2n} + d_{15}d_{31}{}^{-1}S_{12n} \tag{1.8}$$

The electrostatics equations

$$dD_{1n}/d\xi - nD_{2n} = 0 \tag{1.9}$$

$$E_{1n} = -d\psi_n/d\xi, \quad E_{2n} = -n\psi_n \tag{1.10}$$

The strain-displacement formulas

$$\varepsilon_{1n} = du_n/d\xi, \quad \varepsilon_{2n} = -nv_n - qw_n \tag{1.11}$$

$$\omega_n = dv_n/d\xi + nu_n \tag{1.12}$$

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